

A NEW HYPOELLIPTIC OPERATOR ON ALMOST CR MANIFOLDS

RAPHAËL PONGE

ABSTRACT. The aim of this paper is to present the construction, out of the Kohn-Rossi complex, of a new hypoelliptic operator Q_L on almost CR manifolds equipped with a real structure. The operator acts on all (p, q) -forms, but when restricted to $(p, 0)$ -forms and (p, n) -forms it is a sum of squares up to sign factor and lower order terms. Therefore, only a finite type condition is needed to have hypoellipticity on those forms. However, outside these forms Q_L may fail to be hypoelliptic, as it is shown in the example of the Heisenberg group \mathbb{H}^5 . We also look at the Fredholm properties of Q_L and show that the corresponding Fredholm index is zero.

INTRODUCTION

Homogeneity reasons prevent natural operators on CR manifolds to be elliptic, but they can be hypoelliptic in various other guises. An important example is the Kohn Laplacian: under suitable geometric conditions (i.e. $Y(q)$ -condition) the Kohn Laplacian is maximal hypoelliptic and hypoelliptic with a gain of 1 derivative (see [Koh1], [FS], [BG]), but in general it may have rather subtle hypoelliptic properties (see [Ch1], [Ch2], [FK], [FKM], [Ko], [Koh2], [NS]).

The aim of this paper is to present the construction of a new hypoelliptic operator on almost CR manifolds, that is, manifolds M together with a subbundle $H \subset TM$ which is equipped with an almost complex structure $J \in \text{End } H$, $J^2 = -1$. This construction is partly inspired by the second order signature operator of Connes-Moscovici [CM] and an earlier version on 3-dimensional contact manifolds was presented in [Po].

In order to construct our operator it is crucial to further assume that the horizontal subbundle H admits a real structure, i.e., there exists a subbundle $L \subset H$ so that $H = L \oplus JL$. This implies the vanishing of the first Chern class of H , so an almost CR manifold does not admit a real structure in general. However, as explained in Section 1, there is a handful of interesting examples of CR manifolds which do admit a real structure. Among these are real hypersurfaces of \mathbb{C}^{n+1} that are rigid in the sense of [BRT], nilpotent Lie groups and CR nilmanifolds, some CR symmetric spaces in the sense of [KZ], and contact manifolds equipped with a Legendrian subbundle, including circle bundles associated to the geometric quantization of symplectic manifolds.

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The existence of a real structure allows us to define a chirality operator which is analogous to the Hodge \star -operator and maps (p, q) -forms to $(n - p, n - q)$ -forms (where n is the complex dimension of $\dim H$). We then can define a second order differential operator by letting

$$Q_L = (\bar{\partial}_H^* \bar{\partial}_H - \bar{\partial}_H \bar{\partial}_H^*) - \gamma(\bar{\partial}_H^* \bar{\partial}_H - \bar{\partial}_H \bar{\partial}_H^*)\gamma,$$

where $\bar{\partial}_H$ is the horizontal $\bar{\partial}$ -operator of Kohn-Rossi ([KR], [Koh1]). This operators acts on (p, q) -forms and anticommutes with the chirality operator γ .

When restricted to $(p, 0)$ -forms and (p, n) -forms Q_L agrees with a sum of squares up to sign factor and lower order terms (see Proposition 3.2). Therefore, whenever M has finite type, on these forms Q_L is maximal hypoelliptic, which in this context implies that the operator is hypoelliptic with gain of one derivative (see Section 4). In fact, it even admits a parametrix in the class of singular-integral operators of Rotschild-Stein [RS] and, when $\text{codim } H = 1$, it further has a parametrix in the Heisenberg calculus of Beals-Greiner [BG] and Taylor [Ta]. Notice that in order for all these properties to hold only the finite condition is needed. In particular, when $\text{codim } H = 1$ we may allow (M, H) to be weakly pseudoconvex.

The hypoelliptic properties of Q_L contrast with that of the Kohn Laplacian. For instance, on strictly pseudoconvex CR manifolds Q_L is hypoelliptic precisely in bidegrees where the Kohn Laplacian is not. In particular, in dimension 3 the operator Q_L is hypoelliptic in every bidegree, while the Kohn Laplacian is hypoelliptic in none. In addition, on weakly pseudoconvex CR manifolds that are not strictly pseudoconvex Q_L may be maximal hypoelliptic, while the Kohn Laplacian may not.

On the other hand, outside $(p, 0)$ -forms and $(n, 0)$ -forms the operator Q_L may fail to be hypoelliptic. This fact is illustrated in Section 5, where we look at the operator Q_L on the 5-dimensional Heisenberg group \mathbb{H}^5 . In this setting we explicitly construct a $(0, 1)$ -form which annihilates Q_L but is singular at the origin. This shows that Q_L is not hypoelliptic on $(0, 1)$ -forms. Similar arguments also show that Q_L is not hypoelliptic on $(1, 1)$ -forms or on $(2, 1)$ -forms either. Therefore, in the case of \mathbb{H}^5 this is only on $(p, 0)$ -forms and (p, n) -forms that Q_L is hypoelliptic.

Finally, a long standing problem in CR and contact geometry is to prove an analogue of the Atiyah-Singer index formula in those geometric settings. Part of the difficulty is that for CR and contact manifolds a horizontal version of the Dirac operator construction does not provide us with a hypoelliptic operator. For instance, the CR analogue of the $\bar{\partial} + \bar{\partial}^*$ -operator is the operator $\bar{\partial}_H + \bar{\partial}_H^*$ whose square is the Kohn Laplacian, hence is not hypoelliptic on all $(0, q)$ -forms. Therefore, there is a need to find an alternative to the Dirac construction.

In some sense the operator Q_L provides us with such an alternative. This operator is hypoelliptic and it anticommutes with an analogue of the Hodge \star -operator, so it is very much like a signature operator. The hypoellipticity of Q_L on $(p, 0)$ -forms and (p, n) -forms implies that on those forms Q_L is Fredholm, so we can associate to Q_L a Fredholm index (see Section 6). However, this index can be shown to be zero (see Proposition 6.1), so it does not have any special geometric meaning.

This paper is organized as follows. In Section 1, we present the main definitions and examples regarding real structures on almost CR manifolds. In Section 2, we construct the chirality operator γ mentioned above. In Section 3, we construct the operator Q_L and we derive a local expression which shows that on $(p, 0)$ -forms and (p, n) -forms Q_L is a sum of squares up to sign factor and lower order terms. In

Section 4, we study the hypoellipticity properties of Q_L on $(p, 0)$ -forms and (p, n) -forms. In Section 5, we look at the operator Q_L on the Heisenberg group \mathbb{H}^5 and illustrate on this example the fact that Q_L may fail to be hypoelliptic outside forms of bidegree $(p, 0)$ and (p, n) . In the last section, Section 6, we study the Fredholm properties of the operator Q_L and show that its index is actually zero.

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1. REAL STRUCTURES ON CR MANIFOLDS

Let M be an almost CR manifold, i.e., M is equipped with a subbundle $H \subset TM$ carrying an almost complex structure $J \in C^\infty(M, H)$, $J^2 = -1$. This gives rise to a CR structure when J is integrable, i.e., the subbundle $T_{1,0} := \ker(J + i) \subset T_{\mathbb{C}}M$ is integrable in Frobenius' sense.

We don't assume H to have codimension 1, that is, (M, H) need not be of hypersurface type. In any case $n := \frac{1}{2} \dim H$ is an integer, called the *CR dimension* of M .

In addition, we shall say that M is of *finite type*, when Hörmander's bracket condition is satisfied, i.e., at every point TM is spanned by successive Lie brackets $[X_1, [X_2, [\dots, [X_m, \dots]]]$ of vectors fields with values in H .

Definition 1.1. A real structure on H is given by the datum of a rank n real subbundle $L \subset H$ such that

$$(1.1) \quad H = L \oplus \mathcal{J}L.$$

If L is real structure on H , then the decomposition (1.1) yields an involution $X \rightarrow \underline{X}$ on the fibers of H defined by

$$(1.2) \quad \underline{X_1 + \mathcal{J}Y_1} = X_1 - \mathcal{J}Y_1 \quad \forall X_1, Y_1 \in C^\infty(M, L).$$

Notice that $L = \ker(\underline{} - 1)$ and $\mathcal{J}L = \ker(\underline{} + 1)$. Conversely, if ι is an involutive section of $\text{End}_{\mathbb{R}} H$ anticommuting with \mathcal{J} , then the subbundle $L := \ker(\iota - 1)$ defines a real structure on H .

Let us now look at some examples of real structures.

A. Rigid real hypersurfaces. Let us denote (z^1, \dots, z^n, w) the complex coordinates on \mathbb{C}^{n+1} . In terms of real and imaginary parts we shall write $z^j = x^j + iy^j$, $j = 1, \dots, n$, and $w = u + iv$. Consider a real hypersurface of the form,

$$(1.3) \quad M = \{v = F(z, \bar{z})\},$$

where $F(z, \bar{z})$ is some real-valued function. In the terminology of [BRT] such a hypersurface is said to be *rigid*. Examples of such hypersurfaces are given by the hyperquadrics $Q_{p,q}^{2n+1} := \{v = \sum_{j=1}^p |z^j|^2 - \sum_{j=p+1}^q |z^j|^2\}$, $p + q = n$.

We equip M with the CR structure induced by the complex structure of \mathbb{C}^{n+1} , i.e., the differential J_0 of the multiplication by i on $T\mathbb{C}^{n+1}$. Therefore we have $H = TM \cap J_0(TM)$ and the complex structure of H is just $J = J_0|_M$. Then the CR tangent bundle $T_{1,0} := \ker(J - i)$ agrees with $T^{1,0}\mathbb{C}^{n+1} \cap T_{\mathbb{C}}M$. In particular, a global frame of $T_{1,0}$ is given by the vector fields,

$$(1.4) \quad Z_j = \frac{\partial}{\partial z^j} + i \partial_{z^j} F(z, \bar{z}) \frac{\partial}{\partial w} \quad j = 1, \dots, n.$$

For $j = 1, \dots, n$, set $Z_j = X_j - iY_j$, where X_j and Y_j denote the real and imaginary parts of Z_j , i.e.,

$$(1.5) \quad X_j = \frac{\partial}{\partial x^j} + \partial_{y^j} F \frac{\partial}{\partial u} + \partial_{x^j} F \frac{\partial}{\partial v}, \quad Y_j = \frac{\partial}{\partial y^j} - \partial_{x^j} F \frac{\partial}{\partial u} + \partial_{y^j} F \frac{\partial}{\partial v}.$$

Then the vector fields X_1, \dots, X_n and Y_1, \dots, Y_n form a frame of H such that $JX_j = Y_j$. Therefore, if we let L denote the subbundle spanned by X_1, \dots, X_n then $H = L \oplus JL$, that is, L defines a real structure on H .

B. Nilpotent Lie groups and CR nilmanifolds. Let \mathbb{H}^{2n+1} denote the $(2n+1)$ -dimensional Heisenberg group. We realize \mathbb{H}^{2n+1} as $\mathbb{R} \times \mathbb{R}^{2n}$ equipped with the group law,

$$(1.6) \quad x.y = (x^0 + y^0 + \sum_{1 \leq j \leq n} (x^{n+j}y^j - x^jy^{n+j}), x^1 + y^1, \dots, x^{2n} + y^{2n}).$$

This group law is homogeneous with respect to the parabolic dilations,

$$(1.7) \quad t.(x^0, x^1, \dots, x^{2n}) = (t^2x^0, tx^1, \dots, tx^{2n}) \quad t > 0.$$

Identifying the Lie algebra \mathfrak{h}^{2n+1} of \mathbb{H}^{2n+1} with the Lie algebra of left-invariant vector fields, a basis for \mathfrak{h}^{2n+1} is provided by the left-invariant vector-fields,

$$(1.8) \quad X_0 = \frac{\partial}{\partial x^0}, \quad X_j = \frac{\partial}{\partial x^j} + x^{n+j} \frac{\partial}{\partial x^0}, \quad X_{n+j} = \frac{\partial}{\partial x^{n+j}} - x^j \frac{\partial}{\partial x^0},$$

where j ranges from 1 to n . In particular, for $j, k = 1, \dots, n$ and $k \neq j$ we have the relations $[X_j, X_{n+k}] = -2\delta_{jk}X_0$ and $[X_0, X_j] = [X_j, X_k] = [X_{n+j}, X_{n+k}] = 0$. Notice also that with respect to the dilations (1.7) the vector fields X_0 is homogeneous of degree -2 , while X_1, \dots, X_{2n} are homogeneous of degree -1 .

Let H be the subbundle spanned by X_1, \dots, X_{2n} . We endow H with the complex structure \mathcal{J} such that $\mathcal{J}X_j = X_{n+j}$ and $\mathcal{J}X_{n+j} = -X_j$. This defines a homogeneous left-invariant CR structure on \mathbb{H}^{2n+1} . A left-invariant real structure on H is then provided by the subbundle L spanned by X_1, \dots, X_n .

More generally, let G be a real nilpotent Lie group which is stratified, in the sense that its Lie algebra $\mathfrak{g} := T_0G$ admits a grading by vector subspaces,

$$(1.9) \quad \mathfrak{g} = \mathfrak{h}_1 \oplus \dots \oplus \mathfrak{h}_k,$$

such that $\mathfrak{h}_j := [\mathfrak{h}_1, \mathfrak{h}_{j-1}]$, $j = 2, \dots, k$. Set $\mathfrak{h} = \mathfrak{h}_1$ and assume that $\dim_{\mathbb{R}} \mathfrak{h}$ is even, say $\dim_{\mathbb{R}} \mathfrak{h} = 2n$. As a real vector space \mathfrak{h} is isomorphic to \mathbb{C}^n , so it admits a complex structure \mathcal{J}_0 . Let H be the subbundle of left-invariant vector fields such that $H|_{x=0} = \mathfrak{h}$ and let \mathcal{J} be the almost complex structure on H such that $\mathcal{J}|_{x=0} = \mathcal{J}_0$. Then H and \mathcal{J} define a left-invariant almost CR structure on G .

Let \mathfrak{l} be an n -dimensional real subspace of \mathfrak{h} such that $\mathfrak{h} = \mathfrak{l} \oplus \mathcal{J}_0\mathfrak{l}$, and let L denote the subbundle of left-invariant vector fields such that $L|_{x=0} = \mathfrak{l}$. Then $H = L \oplus \mathcal{J}L$, so L gives rise to a left-invariant real structure on H .

Let $\Gamma \subset G$ be a lattice, i.e., a discrete cocompact subgroup of G . Then $M := \Gamma \backslash M$ is a compact nilmanifold. Since H and \mathcal{J} are left-invariant, and Γ acts on left, they descend to M , hence define a natural CR structure on M . Similarly L descends to a vector bundle on M and thereby gives rise to a real structure on M .

C. *CR Symmetric Spaces.* Let G be a connected Lie group with Lie algebra \mathfrak{g} and let $j \in G$ have order 4. Let τ be the automorphism $\tau(x) = jxj^{-1}$, $x \in G_0$, and let G_τ denote the fixed point group of τ . Set $s = j^2$ and $\sigma = \tau^2$, so that σ is an involutive automorphism of G_0 .

Let K be a compact subgroup of G_τ with Lie algebra \mathfrak{k} and let \mathfrak{l} be an $\text{Ad}(K)$ -invariant subspace of $\ker(\text{Ad}(s) + 1)$ such that $\mathfrak{l} \cap \text{Ad}(j)\mathfrak{l} = \{0\}$. Set $\mathfrak{h} = \mathfrak{l} \oplus \text{Ad}(j)\mathfrak{l}$; this is an $\text{Ad}(K)$ -invariant subspace. Let \mathfrak{a} denote the Lie algebra generated by \mathfrak{h} and let us further assume that $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}$. In addition, since K is compact and \mathfrak{h} is $\text{Ad}(K)$ -invariant, there is a subspace \mathfrak{p} of \mathfrak{g} containing \mathfrak{h} such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Under these conditions $M := G/K$ is a CR symmetric space in the sense of Kaup-Zaitsev (see [KZ, Sect. 6]). Let $o \in M$ denote the class of 1. Then $T_o M$ is isomorphic to \mathfrak{p} and under this identification there are a unique G -invariant bundle $H \subset TM$ and a unique G -invariant almost complex \mathcal{J} structure on H such that $H_o = \mathfrak{h}$ and $\mathcal{J}_o = \text{Ad}(j)|_{\mathfrak{h}}$. In addition, the condition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}$, where \mathfrak{a} is the Lie algebra of \mathfrak{h} , insures us that this CR structure is of finite type (cf. [KZ, Prop. 6.2]).

Let L denote the unique G -invariant subbundle of H such that $L_o = \mathfrak{l}$, then we have $H = L \oplus JL$. Therefore L defines a G -invariant real structure on M .

The above construction can be illustrated by the following example. We take $G = SU(n)$, so that the Lie algebra $\mathfrak{g} = \mathfrak{su}(n)$ consists of trace-free skew-Hermitian matrices. In addition, we let p and q be positive integers so that $n = p + q$. Identifying \mathbb{C}^n with $\mathbb{C}^p \oplus \mathbb{C}^q$ we shall write $n \times n$ -matrices as block matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Set $j := \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$. This is a 4th-order element of $SU(n)$ and we have $s := j^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus,

$$(1.10) \quad \tau \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = \text{Ad}(j) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & ib \\ -ic & d \end{pmatrix},$$

$$(1.11) \quad \sigma \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = \text{Ad}(s) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

In particular, we have the splitting $\mathfrak{su}(n) = \mathfrak{m}_+ \oplus \mathfrak{m}_-$, where

$$(1.12) \quad \mathfrak{m}_+ := \ker(\text{Ad}(s) - 1) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}; a \in \mathfrak{u}(p), d \in \mathfrak{u}(q), \text{Tr } a + \text{Tr } d = 0 \right\},$$

$$(1.13) \quad \mathfrak{m}_- := \ker(\text{Ad}(s) + 1) = \left\{ \begin{pmatrix} 0 & b \\ -b^* & 0 \end{pmatrix} \right\}.$$

Set $K := SO(p) \times SO(q) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}; a \in SO(p), d \in SO(q) \right\}$. This is a (compact) subgroup of G_τ . Define $\mathfrak{l} := \mathfrak{m}_- \cap M_n(\mathbb{R})$, i.e.,

$$(1.14) \quad \mathfrak{l} = \left\{ \begin{pmatrix} 0 & b \\ -b^t & 0 \end{pmatrix}; b \in M_{p,q}(\mathbb{R}) \right\}.$$

Then \mathfrak{l} is an $\text{Ad}(K)$ -invariant subspace such that $\mathfrak{l} \cap \text{Ad}(j)\mathfrak{l} = \{0\}$. In fact, we have $\mathfrak{h} := \mathfrak{l} \oplus \text{Ad}(j)\mathfrak{l} = \mathfrak{m}_-$ and, using elementary matrices, it is not difficult to check that $[\mathfrak{m}_-, \mathfrak{m}_-] = \mathfrak{m}_+$. Therefore \mathfrak{g} agrees with the Lie algebra generated by \mathfrak{h} . It then follows that $SU(n)/(SO(p) \times SO(q))$ is a CR symmetric space of finite type with a $SU(n)$ -invariant real structure defined by \mathfrak{l} .

D. *Contact manifolds, Legendrian subbundles and geometric quantization.* Assume that (M^{2n+1}, H) is an orientable contact manifold, i.e., H is the annihilator of a globally defined contact form θ on M . Let \mathcal{J} be an almost complex structure on H which is calibrated with respect to $d\theta|_H$, i.e., $g_{\theta, \mathcal{J}}(X, Y) := d\theta(X, \mathcal{J}Y)$ is a positive-definite metric on H (since M is orientable such an almost complex structure always exists). In particular (H, \mathcal{J}) defines an almost CR structure on M .

Let L be a Legendrian subbundle of H , i.e., L is a maximal isotropic subbundle of $d\theta|_H$. If X and Y are sections of L , then we have

$$(1.15) \quad g_{\theta, \mathcal{J}}(X, \mathcal{J}Y) = d\theta(X, \mathcal{J}^2Y) = -d\theta(X, Y) = \theta([X, Y]) = 0.$$

This implies the orthogonal decomposition $H = L \oplus \mathcal{J}L$, so L defines a real structure independently of the choice of \mathcal{J} .

Conversely, suppose that L is a subbundle H of rank n such that L and $\mathcal{J}L$ are orthogonal. As in (1.15) if X and Y are sections of L , then

$$(1.16) \quad d\theta(X, Y) = -\theta([X, Y]) = -g_{\theta, \mathcal{J}}(X, \mathcal{J}Y) = 0.$$

Hence L is a Legendrian subbundle of H .

As we shall now recall examples of contact manifolds equipped with a Legendrian subbundle naturally occur in the context of the geometric quantization of symplectic manifolds.

Let (X^{2n}, ω) be a symplectic manifold which is *prequantizable*, i.e., the cohomology class of $\frac{1}{2\pi}\omega$ is integral (see, e.g., [AE], [Wo]). Then there exists a Hermitian line bundle (L, h) on X with a metric connection ∇^L with curvature $F^L = -i\omega$. Let L^* denote the dual line bundle with unit sphere bundle $S^*(L) := \{\xi \in L^*; h(\xi, \xi) = 1\}$. Recall that the connection 1-form of the dual connection ∇^{L^*} makes sense as a globally defined real 1-form on the total space M of $S(L^*)$. This can be seen as follows.

Let ξ be a local section of $S(L^*)$. Regarding ξ as a non-zero section of L^* it defines a local trivialization of L^* with respect to which we have $\nabla^{L^*} = d + i\alpha$, where $\alpha := -ih(\xi, \nabla^{L^*}\xi)$. Notice that, as ∇^{L^*} is a metric connection, α must be a real 1-form. Let λ denote the local fiber coordinate on M defined by ξ and consider the *real* 1-form defined by

$$(1.17) \quad \theta := p^*\alpha - i\lambda^{-1}d\lambda,$$

where $p : M \rightarrow X$ is the fibration of M over X .

Let ξ' be another local section of $S(L^*)$ and set $\alpha' := -ih(\xi', \nabla^{L^*}\xi')$ and $\xi' = \mu\xi$, where μ is an S^1 -valued function. Then $\alpha' = \alpha - i\mu^{-1}d\mu$. The local fiber coordinate on M defined by ξ' is $\lambda' = (p^*\mu)^{-1}\lambda$, so the 1-form (1.17) corresponding to ξ' is

$$(1.18) \quad \begin{aligned} \theta' &= p^*\alpha' - i\lambda'^{-1}d\lambda' = p^*[\alpha - i\mu^{-1}d\mu] - i[(p^*\mu)^{-1}\lambda]^{-1}d[(p^*\mu)^{-1}\lambda] \\ &= p^*\alpha - i\lambda^{-1}d\lambda = \theta. \end{aligned}$$

This shows that the 1-form θ in (1.17) does not depend on the choice of the local section ξ , so it makes sense globally on M .

Let $H = \ker \theta$ and let $V := \ker dp \subset TM$ be the vertical bundle of M . It follows from (1.17) that $H \cap V = \{0\}$, so dp induces an isomorphism from H onto TX . Using (1.17) we also see that locally $d\theta = p^*d\alpha$. By assumption we have $id\alpha = F^{L^*} = -F^L = i\omega$, so $d\theta = p^*\omega$. Since dp induces an isomorphism from H onto TX , it follows that $d\theta$ is non-degenerate on H , i.e., θ is a contact form on H .

Let us further assume that (X, ω) is *quantizable* in the sense that it admits a Lagrangian subbundle, i.e., a subbundle $\Lambda \subset TX$ which is maximal isotropic for ω (see, e.g., [AE], [Wo]). Then the subbundle $L := p^*\Lambda \cap H$ is maximal isotropic for $d\theta$, i.e., L is a Legendrian subbundle of M . Therefore, we see that the quantization of a symplectic manifold naturally gives rise to a contact manifold (M, θ) equipped with a Legendrian subbundle.

2. REAL STRUCTURE AND CHIRALITY OPERATOR

Throughout the rest of the paper we let (M, H) be an almost CR manifold of CR dimension n . We also assume H to have a real structure L , that is,

$$(2.1) \quad H = L \oplus \mathcal{J}L,$$

where \mathcal{J} denotes the almost complex structure of H .

In addition, we endow L with a Riemannian metric g_L . Extending g_L to be zero on $\mathcal{J}L \times H$ and $H \times \mathcal{J}L$, we endow H with the Riemannian metric,

$$(2.2) \quad g_H := g_L(\cdot, \cdot) + g_L(\mathcal{J}\cdot, \mathcal{J}\cdot).$$

With respect to this metric \mathcal{J} becomes an isometry and the splitting (1.1) becomes orthogonal. If (M, H) is of finite type, then we see that (H, g_H) defines a sub-Riemannian structure on M compatible with its the almost CR structure.

We fix a choice of supplement \mathcal{N} of H in TM . This allows us to identify H^* with the annihilator of \mathcal{N} in T^*M . We also set $T_{1,0} = \ker(\mathcal{J} + i)$ and $T_{0,1} = \ker(\mathcal{J} - i)$. Notice that $\overline{T_{1,0}} = T_{0,1}$. Moreover, extending g_H into a Hermitian metric on $T_{\mathbb{C}}M$ we get the orthogonal decomposition,

$$(2.3) \quad H \otimes \mathbb{C} = T_{1,0} \oplus T_{0,1}.$$

For $p, q = 0, \dots, n$ we let $\Lambda^{p,q} := (\Lambda^{1,0})^p \wedge (\Lambda^{0,1})^q$ denote the bundle of (p, q) -covectors, where $\Lambda^{1,0}$ and $\Lambda^{0,1}$ are the respective annihilators in $T_{\mathbb{C}}^*M$ of the subbundles $T_{0,1} \oplus (\mathcal{N} \otimes \mathbb{C})$ and $T_{1,0} \oplus (\mathcal{N} \otimes \mathbb{C})$. We then have the orthogonal splitting,

$$(2.4) \quad \Lambda_{\mathbb{C}}^* H^* = \bigoplus_{p,q=0}^n \Lambda^{p,q}.$$

We shall now turn the bundle of (p, q) -covectors into a super-bundle by equipping it with a suitable chirality operator. To define the latter we shall make use of the real structure of H . To this end we extend the involution (1.2) into the antilinear involution on $H \otimes \mathbb{C}$ defined by

$$(2.5) \quad \underline{X + iY} = \underline{X} - i\underline{Y} \quad \forall X, Y \in C^\infty(M, H).$$

This involution preserves both $T_{1,0}$ and $T_{0,1}$. Therefore, by duality it gives rise to an antilinear involution of $\Lambda^{*,*}$ preserving the bidegree. As we shall see the latter property will be crucial in the construction of the operator Q_L in the next section.

Let $v_H(x)$ be the volume form of g_H (seen as a section of $\Lambda^{n,n}$), and let g_H^* denote the Hermitian metric on $\Lambda^{*,*}$ induced by g_H . The operator $\star : \Lambda^{*,*} \rightarrow \Lambda^{n-*, n-*}$ is uniquely determined by the formula,

$$(2.6) \quad \beta \wedge \underline{\star \alpha} = g_H^*(\beta, \alpha) v_H(x) \quad \forall \alpha, \beta \in C^\infty(M, \Lambda^{p,q}).$$

Let X_1, \dots, X_n be an orthonormal frame of L . Since the splitting $H = L \oplus \mathcal{J}L$ is orthogonal we see that $\{X_j, \mathcal{J}X_j\}$ is an orthonormal frame of H . For $j = 1, \dots, n$ we set $Z_j = \frac{1}{\sqrt{2}}(X_j - i\mathcal{J}X_j)$ and $Z_{\bar{j}} = \frac{1}{\sqrt{2}}(X_j + i\mathcal{J}X_j)$. Then $\{Z_j\}$ and $\{Z_{\bar{j}}\}$

are orthonormal frames of $T_{1,0}$ and $H \otimes \mathbb{C}$ respectively. Any orthonormal frame of $H \otimes \mathbb{C}$ obtained by a similar process is said to be *admissible*. Notice that the invariance of L under the involution (1.2) imply that $\underline{Z_j} = Z_j$ and $\underline{Z_{\bar{j}}} = Z_{\bar{j}}$.

Let $\{\theta^j, \theta^{\bar{j}}\}$ be the coframe of $H^* \otimes \mathbb{C}$ dual to $\{Z_j, Z_{\bar{j}}\}$. For any ordered subsets $J = \{j_1, \dots, j_p\}$ and $K = \{k_1, \dots, k_q\}$ of $\{1, \dots, n\}$ with $j_1 < \dots < j_p$ and $k_1 < \dots < k_q$ we set $\theta^{J, \bar{K}} := \theta^{j_1} \wedge \dots \wedge \theta^{j_p} \wedge \theta^{\bar{k}_1} \wedge \dots \wedge \theta^{\bar{k}_q}$. Then $\{\theta^{J, \bar{K}}\}$ is an orthonormal coframe of $\Lambda^{*,*}$.

If $J = \{j_1, \dots, j_p\}$ is an ordered subset of $\{1, \dots, n\}$ with ordered complement $J^c = \{j'_1, \dots, j'_{n-p}\}$, then we let $\varepsilon(J, J^c)$ denote the signature of the permutation

$$(2.7) \quad (j_1, \dots, j_p, j'_1, \dots, j'_{n-p}) \longrightarrow (1, \dots, n).$$

Notice that we always have $\varepsilon(J, J^c)\varepsilon(J^c, J) = (-1)^{p(n-p)}$, for $\varepsilon(J, J^c)\varepsilon(J^c, J)$ is the signature of the permutation $(j_1, \dots, j_p, j'_1, \dots, j'_{n-p}) \rightarrow (j'_1, \dots, j'_{n-p}, j_1, \dots, j_p)$.

Lemma 2.1. 1) We have

$$(2.8) \quad \star \theta^{J, \bar{K}} = i^n (-1)^{\frac{n(n-1)}{2} + q(n-p)} \varepsilon(J, J^c) \varepsilon(K, K^c) \theta^{J^c, \bar{K}^c}.$$

2) On $\Lambda^{p,q}$ we have

$$(2.9) \quad \star^2 = (-1)^{n+p+q}.$$

Proof. Set $\theta^{n, \bar{n}} := \theta^1 \wedge \dots \wedge \theta^n \wedge \theta^{\bar{1}} \wedge \dots \wedge \theta^{\bar{n}}$. Since H is oriented by means of its almost complex structure, locally we have

$$(2.10) \quad v_H(x) = i^n \theta^1 \wedge \theta^{\bar{1}} \wedge \dots \wedge \theta^n \wedge \theta^{\bar{n}} = i^n (-1)^{\frac{n(n-1)}{2}} \theta^{n, \bar{n}}.$$

Therefore Eq. (2.6) can be rewritten as

$$(2.11) \quad \beta \wedge \star \alpha = i^n (-1)^{\frac{n(n-1)}{2}} g_H^*(\beta, \alpha) \theta^{n, \bar{n}} \quad \forall \alpha, \beta \in C^\infty(M, \Lambda^{p,q}).$$

Let J_0 and K_0 be ordered subsets of $\{1, \dots, n\}$ of respective lengths p and q , and set $\omega := \star \theta^{J_0, \bar{K}_0} = \sum \lambda_{J, \bar{K}} \theta^{J, \bar{K}}$. The (n, n) -component of $\underline{\omega} \wedge \Theta^{J, \bar{K}}$ is equal to $\pm \lambda_{J, \bar{K}^c} \theta^{n, \bar{n}}$, so from (2.11) we see that $\lambda_{J^c, \bar{K}^c} = 0$ unless $J = J_0^c$ and $K = K_0^c$, i.e., we have $\omega = \lambda_{J_0^c, \bar{K}_0^c} \theta^{J_0^c, \bar{K}_0^c}$. In particular, from (2.11) we get

$$(2.12) \quad i^n (-1)^{\frac{n(n-1)}{2}} \theta^{n, \bar{n}} = \theta^{J_0, \bar{K}_0} \wedge \underline{\omega} = \overline{\lambda_{J_0^c, \bar{K}_0^c}} \theta^{J_0, \bar{K}_0} \wedge \theta^{J_0^c, \bar{K}_0^c}.$$

Next, upon writing $\theta^{J_0, \bar{K}_0} = \theta^{J_0, \bar{0}} \wedge \theta^{0, \bar{K}_0}$ and $\theta^{J_0^c, \bar{K}_0^c} = \theta^{J_0^c, \bar{0}} \wedge \theta^{0, \bar{K}_0^c}$ we get

$$(2.13) \quad \begin{aligned} \theta^{J_0, \bar{K}_0} \wedge \theta^{J_0^c, \bar{K}_0^c} &= (-1)^{q(n-p)} \theta^{J_0, \bar{0}} \wedge \theta^{J_0^c, \bar{0}} \wedge \theta^{0, \bar{K}_0} \wedge \theta^{0, \bar{K}_0^c} \\ &= (-1)^{q(n-p)} \varepsilon(J_0, J_0^c) \varepsilon(K_0, K_0^c) \theta^{n, \bar{n}}. \end{aligned}$$

Comparing this with (2.12) yields $\lambda_{J_0^c, \bar{K}_0^c} = i^n (-1)^{\frac{n(n-1)}{2} + q(n-p)} \varepsilon(J_0, J_0^c) \varepsilon(K_0, K_0^c)$. Thus,

$$(2.14) \quad \star \theta^{J_0, \bar{K}_0} = i^n (-1)^{\frac{n(n-1)}{2} + q(n-p)} \varepsilon(J_0, J_0^c) \varepsilon(K_0, K_0^c) \theta^{J_0^c, \bar{K}_0^c}.$$

Now, using (2.8) we see that $\star^2 \theta^{J_0, \bar{K}_0}$ is equal to

$$(2.15) \quad (-1)^{n+q(n-p)+(n-q)p} \varepsilon(J_0^c, J_0) \varepsilon(K_0^c, K_0) \varepsilon(J_0, J_0^c) \varepsilon(K_0, K_0^c) \theta^{J_0, \bar{K}_0}.$$

Since $\varepsilon(J_0^c, J_0) = (-1)^{(n-p)p} \varepsilon(J_0, J_0^c)$ and $\varepsilon(K_0^c, K_0) = (-1)^{(n-q)q} \varepsilon(K_0, K_0^c)$ we get $\star^2 \theta^{J_0, K_0^c} = (-1)^{n+N} \theta^{J_0, K_0^c}$, with

$$(2.16) \quad N = q(n-p) + (n-q)p + (n-p)p + (n-q)q = (2n-p-q)(p+q) \\ = (p+q)^2 = p+q \pmod{2}.$$

Hence $\star^2 \theta^{J_0, K_0^c} = (-1)^{n+p+q} \theta^{J_0, K_0^c}$. This proves that $\star^2 = (-1)^{n+p+q}$ on $\Lambda^{p,q}$. \square

Next, the chirality operator $\gamma : \Lambda^{*,*} \rightarrow \Lambda^{*,*}$ is defined by

$$(2.17) \quad \gamma = i^{n+(p+q)^2} \star \quad \text{on } \Lambda^{p,q}.$$

Notice that γ maps $\Lambda^{p,q}$ onto $\Lambda^{n-p, n-q}$.

Lemma 2.2. *The operator γ is a \mathbb{Z}_2 -grading, that is, it satisfies*

$$(2.18) \quad \gamma^2 = 1 \quad \text{and} \quad \gamma^* = \gamma.$$

Proof. First, thanks to Lemma 2.1, on $\Lambda^{p,q}$ we have

$$(2.19) \quad \gamma^2 = i^{2n+(2n-p-q)^2+(p+q)^2} \star^2 = i^{2n+2(p+q)^2} (-1)^{n+p+q} = 1.$$

Next, let $\{\theta^{J, \bar{K}}\}$ be the coframe of $\Lambda^{*,*}$ associated to an admissible frame $\{Z_j, Z_{\bar{j}}\}$ of $H \otimes \mathbb{C}$, and let J and K be ordered subsets of respective lengths p and q . Then by (2.8) and (2.17) we have

$$(2.20) \quad \gamma \theta^{J, \bar{K}} = i^{(p+q)^2} (-1)^{\frac{n(n-1)}{2} + q(n-p)} \varepsilon(J, J^c) \varepsilon(K, K^c) \theta^{J^c, \bar{K}^c}.$$

Since $\{\theta^{J, \bar{K}}\}$ is an orthonormal coframe, this gives

$$(2.21) \quad \gamma^* \theta^{J^c, \bar{K}^c} = (-i)^{(p+q)^2} (-1)^{\frac{n(n-1)}{2} + q(n-p)} \varepsilon(J, J^c) \varepsilon(K, K^c) \theta^{J, \bar{K}}.$$

Thus for $\theta^{J, \bar{K}}$ we obtain

$$(2.22) \quad \gamma^* \theta^{J, \bar{K}} = (-i)^{(2n-p-q)^2} (-1)^{\frac{n(n-1)}{2} + (n-q)p} \varepsilon(J^c, J) \varepsilon(K^c, K) \theta^{J^c, \bar{K}^c}, \\ = (-i)^{(p+q)^2} (-1)^{\frac{n(n-1)}{2} + (n-q)p + p(n-p) + q(n-q)} \varepsilon(J, J^c) \varepsilon(K, K^c) \theta^{J^c, \bar{K}^c}.$$

As by (2.16) we have $(-1)^{(n-q)p + p(n-p) + q(n-q)} = (-1)^{p+q+q(n-p)}$, we get

$$(2.23) \quad \gamma^* \theta^{J, \bar{K}} = i^{(p+q)^2} (-1)^{\frac{n(n-1)}{2} + q(n-p)} \varepsilon(J, J^c) \varepsilon(K, K^c) \theta^{J^c, \bar{K}^c} = \gamma \theta^{J, \bar{K}}.$$

Hence $\gamma^* = \gamma$. The lemma is thus proved. \square

3. THE OPERATOR Q_L

We shall now construct a differential operator Q_L acting on the sections of $\Lambda^{*,*}$ which is supersymmetric in the sense that it anticommutes with the chirality operator γ . Furthermore, in suitable bidegree this operator will be hypoelliptic under the finite type condition alluded to in Section 1.

First, recall that the tangential $\bar{\partial}$ -operator of Kohn-Rossi ([KR], [Koh1]) can be defined as follows. For any $\eta \in C^\infty(M, \Lambda^{p,q})$ its differential uniquely decomposes as

$$(3.1) \quad d\eta = \bar{\partial}_H \eta + \partial_H \eta + \theta \wedge \mathcal{L}_{X_0} \eta,$$

where $\bar{\partial}_H \eta$ (resp. $\partial_H \eta$) is a section of $\Lambda^{p, q+1}$ (resp. $\Lambda^{p+1, q}$). Moreover, when $T_{1,0}$ is integrable $\bar{\partial}_H^2$ vanishes on $(0, q)$ -forms, so that we then get a cochain complex $\bar{\partial}_H : C^\infty(M, \Lambda^{0,*}) \rightarrow C^\infty(M, \Lambda^{0,*+1})$.

The operator Q_L is defined by

$$(3.2) \quad Q_L = (\bar{\partial}_H^* \bar{\partial}_H - \bar{\partial}_H \bar{\partial}_H^*) - \gamma(\bar{\partial}_H^* \bar{\partial}_H - \bar{\partial}_H \bar{\partial}_H^*)\gamma.$$

In order to determine the local expression of Q_L , let $\{Z_j, Z_{\bar{j}}\}$ be an admissible orthonormal frame of $H \otimes \mathbb{C}$, and let $\{\theta^j, \theta^{\bar{j}}\}$ be the associated dual coframe of $H^* \otimes \mathbb{C}$. In addition, we let $\varepsilon(\theta^{\bar{j}})$ denote the exterior multiplication by $\theta^{\bar{j}}$ and let $\iota(\theta^j)$ denote the interior product by θ^j (i.e. $\iota(\theta^j)$ is the contraction of forms by the vector field $Z_{\bar{j}}$).

Lemma 3.1. *For $j, k = 1, \dots, n$ we have*

$$(3.3) \quad \gamma\varepsilon(\theta^{\bar{j}})\gamma = i\iota(\theta^j), \quad \gamma\iota(\theta^j)\gamma = -i\varepsilon(\theta^{\bar{j}}),$$

$$(3.4) \quad \gamma\varepsilon(\theta^{\bar{j}})\iota(\theta^k)\gamma = \iota(\theta^j)\varepsilon(\theta^{\bar{k}}), \quad \gamma\iota(\theta^j)\varepsilon(\theta^{\bar{k}})\gamma = \varepsilon(\theta^{\bar{j}})\iota(\theta^k).$$

Proof. First, since $\gamma^2 = 1$ the equalities $\gamma\varepsilon(\theta^{\bar{j}})\gamma = i\iota(\theta^j)$ and $\gamma\iota(\theta^j)\gamma = -i\varepsilon(\theta^{\bar{j}})$ are equivalent to each other. Moreover, we can deduce from them the equalities (3.4). Therefore, we only have to prove that $\gamma\varepsilon(\theta^{\bar{j}})\gamma = i\iota(\theta^j)$.

If $J = \{j_1, \dots, j_p\}$ is an ordered subset of $\{1, \dots, n\}$ and j is an element of J^c such that $j_k < j < j_{k+1}$ we let $\tilde{\varepsilon}(j, J) = (-1)^k$, so that $\tilde{\varepsilon}(j, J)$ is the signature of the permutation $(j, j_1, \dots, j_p) \rightarrow (j_1, \dots, j_k, j, j_{k+1}, \dots, j_p)$. Then

$$(3.5) \quad \varepsilon(\theta^{\bar{j}})\theta^{J, \bar{K}} = \begin{cases} \tilde{\varepsilon}(j, K)\theta^{J, \overline{K \cup \{j\}}} & \text{if } j \notin K, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we have

$$(3.6) \quad \iota(\theta^{\bar{j}})\theta^{J, \bar{K}} = \begin{cases} \tilde{\varepsilon}(j, K \setminus \{j\})\theta^{J, \overline{K \setminus \{j\}}} & \text{if } j \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Let J and K be ordered subsets of $\{1, \dots, n\}$ of respective lengths p and q . Using (2.20) we get

$$(3.7) \quad \gamma\varepsilon(\theta^{\bar{j}})\gamma\theta^{J, \bar{K}} = i^{(p+q)^2}(-1)^{\frac{n(n-1)}{2}+q(n-p)}\varepsilon(J, J^c)\varepsilon(K, K^c)\gamma\varepsilon(\theta^{\bar{j}})\theta^{J^c, \bar{K}^c}.$$

In particular, we see that $\gamma\varepsilon(\theta^{\bar{j}})\gamma\theta^{J, \bar{K}} = 0$ if $j \notin K$.

Assume now that j is in K . Then using (2.20), (3.5) and (3.7) shows that

$$(3.8) \quad \begin{aligned} \gamma\varepsilon(\theta^{\bar{j}})\gamma\theta^{J, \bar{K}} &= i^{(p+q)^2}(-1)^{\frac{n(n-1)}{2}+q(n-p)}\varepsilon(J, J^c)\varepsilon(K, K^c)\tilde{\varepsilon}(j, K^c)\gamma\theta^{J^c, \overline{K^c \cup \{j\}}} \\ &= \lambda_1 \lambda_2 \lambda_3 \theta^{J, \overline{K \setminus \{j\}}}, \end{aligned}$$

where we have let

$$(3.9) \quad \lambda_1 = i^{(p+q)^2+(2n-p-q+1)^2}, \quad \lambda_2 = (-1)^{q(n-p)+(n-q+1)p}\varepsilon(J, J^c)\varepsilon(J, J^c),$$

$$(3.10) \quad \lambda_3 = \varepsilon(K, K^c)\varepsilon(K^c \cup \{j\}, K \setminus \{j\})\tilde{\varepsilon}(j, K^c).$$

Recall that given any integer m the difference $m^2 - m = m(m-1)$ is always an even number. Thus,

$$(3.11) \quad \lambda_1 = i^{(p+q)^2+(p+q-1)^2} = i^{2[(p+q)^2-(p+q)]+1} = i.$$

Moreover, as $\varepsilon(J, J^c)\varepsilon(J, J^c) = (-1)^{p(n-p)}$ we have

$$(3.12) \quad \lambda_2 = (-1)^{q(n-p)+(n-q)p+p(p-n-p)} = (-1)^{nq-2pq+2np+p-p^2} = (-1)^{nq}.$$

Next, set $K = \{k_1, \dots, k_q\}$ and $K^c \cup \{j\} = \{k'_1, \dots, k'_{n-q+1}\}$. Then we have $j = k_l = k'_l$, for some indices l and l' . By definition $\varepsilon(K, K^c)$ is the signature of the

permutation $(k_1, \dots, k_q, k'_1, \dots, \hat{k}'_{l'}, \dots, k'_{n-q+1}) \rightarrow (1, \dots, n)$. The latter can also be seen the composition of the following permutations,

$$\begin{aligned}
(3.13) \quad & (k_1, \dots, k_q, k'_1, \dots, \hat{k}'_{l'}, \dots, k'_{n-q+1}) \\
& \rightarrow (j, k_1, \dots, \hat{k}_l, \dots, k_q, k'_1, \dots, \hat{k}'_{l'}, \dots, k'_{n-q+1}) \\
& \rightarrow (k_1, \dots, \hat{k}_l, \dots, k_q, j, k'_1, \dots, \hat{k}'_{l'}, \dots, k'_{n-q+1}) \\
& \rightarrow (k_1, \dots, \hat{k}_l, \dots, k_q, k'_1, \dots, k'_{n-q+1}) \rightarrow (1, \dots, n).
\end{aligned}$$

The respective signatures of these permutations are

$$(3.14) \quad \tilde{\varepsilon}(j, K \setminus \{j\}), \quad (-1)^{q-1}, \quad \tilde{\varepsilon}(j, K^c),$$

$$(3.15) \quad \varepsilon(K \setminus \{j\}, K^c \cup \{j\}) = (-1)^{(q-1)(n-q+1)} \varepsilon(K^c \cup \{j\}, K \setminus \{j\}).$$

As $(-1)^{q-1}(-1)^{(q-1)(n-q+1)} = (-1)^{(q-1)(n-q)} = (-1)^{nq-q^2-n+q} = (-1)^{n+nq}$ we deduce that $\varepsilon(K, K^c) = (-1)^{(q-1)(n-q)+\tilde{\varepsilon}(j, K \setminus \{j\})\tilde{\varepsilon}(j, K^c)\varepsilon(K^c \cup \{j\}, K \setminus \{j\})}$. Thus,

$$(3.16) \quad \lambda_3 = (-1)^{n+nq} \varepsilon(j, K \setminus \{j\}).$$

Now, combining (3.8) with (3.11), (3.12) and (3.16) gives

$$(3.17) \quad \gamma \varepsilon(\theta^{\bar{j}}) \gamma \theta^{J, \bar{K}} = i(-1)^n \tilde{\varepsilon}(j, K \setminus \{j\}) \theta^{J, \overline{K \setminus \{j\}}},$$

so using (3.6) we get

$$(3.18) \quad \gamma \varepsilon(\theta^{\bar{j}}) \gamma \theta^{J, \bar{K}} = i(-1)^n \iota(\theta^{\bar{j}}) \theta^{J, \bar{K}}.$$

Since $\iota(\theta^{\bar{j}}) \theta^{J, \bar{K}} = 0$ when $j \notin K$, this proves that $\gamma \varepsilon(\theta^{\bar{j}}) \gamma = i(-1)^n \iota(\theta^{\bar{j}})$. The proof is thus achieved. \square

In the sequel we let $O_H(0)$ denote a general zeroth order differential operator and we let $O_H(1)$ denote a first order differential operator involving only differentiations along $H \otimes \mathbb{C}$. For instance, seen as differential operators, Z_j and $Z_{\bar{j}}$ both are $O_H(1)$, but X_0 is not. Bearing this in mind the following holds.

Proposition 3.2. *In the local trivialization of $\Lambda^{*,*}$ defined by the orthonormal coframe $\{\theta^{J, \bar{K}}\}$ we have*

$$(3.19) \quad Q_L = \sum_{j,k=1}^n \left(\varepsilon(\theta^{\bar{j}}) \iota(\theta^k) - \iota(\theta^j) \varepsilon(\theta^{\bar{k}}) \right) (Z_{\bar{j}} Z_k + Z_j Z_{\bar{k}}) + O_H(1).$$

In particular, in the local trivialization of $\Lambda^{,0} \oplus \Lambda^{*,n}$ we have*

$$(3.20) \quad Q_L = \pm \sum_{j=1}^n (Z_j Z_{\bar{j}} + Z_{\bar{j}} Z_j) + O_H(1),$$

where the sign \pm is $-$ on $\Lambda^{,0}$ and $+$ on $\Lambda^{*,n}$.*

Proof. Set $Q'_L := \bar{\partial}_H^* \bar{\partial}_H - \bar{\partial}_H \bar{\partial}_H^*$. One can check (see, e.g, [BG]) that in the trivialization of $\Lambda^{*,*}$ defined by $\{\theta^{J, \bar{K}}\}$ we have

$$(3.21) \quad \bar{\partial}_H = \sum_{j=1}^n \varepsilon(\theta^{\bar{j}}) Z_{\bar{j}} \quad \text{and} \quad \bar{\partial}_H^* = - \sum_{j=1}^n \iota(\theta^j) Z_j + O_H(0).$$

Therefore,

$$\begin{aligned}
(3.22) \quad Q'_L &= - \sum_{j,k=1}^n \iota(\theta^j) \varepsilon(\theta^{\bar{k}}) Z_j Z_{\bar{k}} + \sum_{j,k=1}^n \varepsilon(\theta^{\bar{j}}) \iota(\theta^k) Z_{\bar{j}} Z_k + O_H(1) \\
&= \sum_{j,k=1}^n \left(\varepsilon(\theta^{\bar{j}}) \iota(\theta^k) Z_{\bar{j}} Z_k - \iota(\theta^j) \varepsilon(\theta^{\bar{k}}) Z_j Z_{\bar{k}} \right) + O_H(1).
\end{aligned}$$

Using Lemma 3.1 we get

$$\begin{aligned}
(3.23) \quad \gamma Q'_L \gamma &= \sum_{j,k=1}^n \left(\gamma \varepsilon(\theta^{\bar{j}}) \iota(\theta^k) \gamma Z_{\bar{j}} Z_k - \gamma \iota(\theta^j) \varepsilon(\theta^{\bar{k}}) \gamma Z_j Z_{\bar{k}} \right) + O_H(1) \\
&= \sum_{j,k=1}^n \left(\iota(\theta^j) \varepsilon(\theta^{\bar{k}}) Z_{\bar{j}} Z_k - \varepsilon(\theta^{\bar{j}}) \iota(\theta^k) Z_j Z_{\bar{k}} \right) + O_H(1).
\end{aligned}$$

Combining this with (3.22) then shows that modulo $O_H(1)$ -terms we have

$$\begin{aligned}
Q_L &= Q'_L - \gamma Q'_L \gamma \\
&= \sum_{j,k=1}^n \left(\varepsilon(\theta^{\bar{j}}) \iota(\theta^k) Z_{\bar{j}} Z_k - \iota(\theta^j) \varepsilon(\theta^{\bar{k}}) Z_j Z_{\bar{k}} - \iota(\theta^j) \varepsilon(\theta^{\bar{k}}) Z_{\bar{j}} Z_k + \varepsilon(\theta^{\bar{j}}) \iota(\theta^k) Z_j Z_{\bar{k}} \right) \\
&= \sum_{j,k=1}^n \left(\varepsilon(\theta^{\bar{j}}) \iota(\theta^k) - \iota(\theta^j) \varepsilon(\theta^{\bar{k}}) \right) (Z_{\bar{j}} Z_k + Z_j Z_{\bar{k}}).
\end{aligned}$$

Now, on $\Lambda^{p,0}$ we have $\varepsilon(\theta^{\bar{j}}) \iota(\theta^k) = 0$ and $\iota(\theta^j) \varepsilon(\theta^{\bar{k}}) = \delta^{\bar{j}k}$. Therefore, on $(p,0)$ -forms we have

$$(3.24) \quad Q_L = \sum_{j,k=1}^n (-\delta^{\bar{j}k}) (Z_{\bar{j}} Z_k + Z_j Z_{\bar{k}}) + O_H(1) = - \sum_{j=1}^n (Z_{\bar{j}} Z_j + Z_j Z_{\bar{j}}) + O_H(1).$$

Similarly, as on $\Lambda^{p,n}$ we have $\varepsilon(\theta^{\bar{j}}) \iota(\theta^k) = \delta^{\bar{j}k}$ and $\iota(\theta^j) \varepsilon(\theta^{\bar{k}}) = 0$, we see that on (p,n) -forms $Q_L = \sum_{j=1}^n (Z_{\bar{j}} Z_j + Z_j Z_{\bar{j}}) + O_H(1)$. The proof is now complete. \square

4. HYPOELLIPTIC PROPERTIES OF Q_L

From now on we assume that M is compact. This assumption is not essential, but it will simplify the exposition of what follows. In fact, all the following results can be localized and, as such, they continue to hold in the non-compact case.

As $\{Z_j\}$ is an admissible frame, we can write $Z_j = \frac{1}{\sqrt{2}}(X_j - iX_{n+j})$, where $\{X_j\}$ is a local orthonormal frame of L and $X_{n+j} := \mathcal{J}X_j$. Then using (3.19) we can check that on $(p,0)$ -forms and (p,n) -forms we have

$$(4.1) \quad Q_L = \pm(X_1^2 + \dots + X_{2n}^2) + O_H(1).$$

This means that, up to sign factor and to an $O_H(1)$ -term, on these forms Q_L is a sum of squares. A well-known result of Hörmander [Hö] then insures us that when (M, H) has finite type such an operator is hypoelliptic with gain of $2/r$ derivatives, where r denotes the minimal number of Lie brackets of vector fields with values of H that are needed to span TM . In other words, for all $s \in \mathbb{R}$, we have

$$(4.2) \quad Q_L u \in L_s^2 \implies u \in L_{s+\frac{2}{r}}^2.$$

Given vector fields X_1, \dots, X_m spanning H at every point, Folland-Stein [FS] and Rothschild-Stein [RS] introduced suitable functional spaces to study sums of squares. Namely, for $k = 0, 1, 2, \dots$ they defined

$$(4.3) \quad S_k^2(M) := \bigcap_{1 \leq l \leq k} \{u \in L^2; X_{i_1} \dots X_{i_l} u \in L^2 \ \forall i_j \in \{1, \dots, m\}\},$$

which is a Hilbert space when endowed with the Hilbert norm,

$$(4.4) \quad \|u\|_{S_k^2} := \left(\sum_{1 \leq l \leq k} \sum_{i_1, \dots, i_l} \|X_{i_1} \dots X_{i_l} u\|^2 \right)^{\frac{1}{2}}, \quad u \in S_k^2(M).$$

These definitions also makes sense for sections of any vector bundle over M .

If P is a differential operator of order m on M , we say that P is *maximal hypoelliptic* if, for all $k \in \mathbb{N}$,

$$(4.5) \quad Pu \in S_k^2 \implies u \in S_{k+m}^2.$$

Rothschild-Stein [RS] proved that if (M, H) has finite type, then a sum of squares is maximal hypoelliptic and we have a continuous inclusion $S_k^2 \subset L_{\frac{k}{r}}^2$. Incidentally, maximal hypoellipticity implies hypoellipticity with gain of $\frac{2}{r}$ -derivatives.

In fact, Rothschild-Stein [RS] and Rothschild-Tartakoff [RT] even constructed parametrices for sum of squares in a suitable class of singular-integral operators. These operators enjoy various regularity properties, including mapping continuously S_k^2 to S_{k+2}^2 . We refer to [RS] for a thorough account on these properties.

Summarizing all this we obtain:

Proposition 4.1. *Assume that (M, H) is of finite type. Then on $(p, 0)$ -forms and (p, n) -forms Q_L is maximal hypoelliptic and admits a parametrix in the class of singular-integral operators of Rothschild-Stein.*

Suppose now that (M, H) is of hypersurface type, i.e., $\text{codim } H = 1$. The Levi form of (M, H) is then defined as the Hermitian form,

$$(4.6) \quad \mathcal{L} : T_{1,0} \times T_{1,0} \longrightarrow T_{\mathbb{C}}M/(H \otimes \mathbb{C})$$

such that, for all sections Z and W of $T_{1,0}$ and for all $x \in M$, we have

$$(4.7) \quad \mathcal{L}_x(Z(x), W(x)) = [Z, \overline{W}](x) \mod H_x \otimes \mathbb{C}.$$

It is not difficult to check that (M, H) is of finite type if and only if \mathcal{L} does not vanish anywhere.

On the other hand, when $\text{codim } H = 1$ Beals-Greiner [BG] (see also [Ta]) constructed a pseudodifferential calculus, the so-called Heisenberg calculus, containing a full symbolic calculus allowing us to *explicitly* construct parametrices for sums of squares, as well as for the Kohn Laplacian under the so-called condition $Y(q)$ (see [BG]). Therefore, we obtain:

Proposition 4.2. *1) If $\text{codim } H = 1$ and \mathcal{L} is non-vanishing, then on $(p, 0)$ -forms and (p, n) -forms Q_L admits a parametrix in the Heisenberg calculus.*

2) If $\dim M = 3$ and \mathcal{L} is non-vanishing, then in every bidegree Q_L is hypoelliptic and admits a parametrix in the Heisenberg calculus.

The hypoellipticity properties of Q_L show a new phenomenon with respect to what happens for the Kohn Laplacian, i.e., the Laplacian of the $\bar{\partial}_H$ -complex,

$$(4.8) \quad \square_H := \bar{\partial}_H^* \bar{\partial}_H + \bar{\partial}_H \bar{\partial}_H^*.$$

For CR manifolds of hypersurface type the invertibility in the Heisenberg calculus' sense of the principal symbol of \square_H on (p, q) -forms is equivalent to the $Y(q)$ -condition of Kohn [Koh1] (see [BG]).

When the CR manifold (M, H) is strictly pseudoconvex the condition $Y(q)$ means that we must have $0 < q < n$. In particular, this excludes all the (p, q) -forms in dimension 3. Thus, in the strictly pseudoconvex case, the operator Q_L has an invertible principal symbol precisely on forms where the Kohn Laplacian has not an invertible principal symbol.

When (M, H) is not strictly pseudoconvex, but is weakly pseudoconvex, then the $Y(q)$ -condition always fails. However, if (M, H) has finite type and the Levi form has comparable eigenvalues then the Kohn Laplacian is hypoelliptic (see [Ko] and the references therein). There also are examples of CR manifolds whose Levi form does not have comparable eigenvalues and for which the Kohn Laplacian still enjoys nice regularity properties (see, e.g., [FKM], [NS]).

In contrast, the hypoellipticity of Q_L on $(p, 0)$ -forms and (p, n) -forms is independent of any convexity property of the Levi form, since the sole non-vanishing of \mathcal{L} is enough to have maximal hypoellipticity.

5. FAILURE OF HYPOELLIPTICITY ON \mathbb{H}^5

In the previous section we saw that when restricted to $(p, 0)$ -forms and (p, n) -forms Q_L is maximal hypoelliptic. In this section we would like to explain that that when restricted to other forms the operator Q_L may fail to be hypoelliptic. To this end we shall look at the example of Q_L on the 5-dimensional Heisenberg group \mathbb{H}^5 acting on $(0, 1)$ -forms.

Notice that the (localized versions) of the notions of hypoellipticity alluded to in the previous section all imply the following usual notion of hypoellipticity:

$$(5.1) \quad Q_L u \in C^\infty \implies u \in C^\infty.$$

For homogeneous left-invariant differential operators on \mathbb{H}^5 (and more generally on nilpotent graded Lie groups) this can be shown to be equivalent to maximal hypoellipticity (see [Fo]). In this section we shall exhibit a $(0, 1)$ -form on \mathbb{H}^5 which is singular at the origin and annihilates Q_L . This will prove that Q_L is not hypoelliptic on $(0, 1)$ -forms.

Throughout this section we will keep on using the notation introduced in Example B of Section 1 to describe the Heisenberg group. Thus \mathbb{H}^5 is $\mathbb{R} \times \mathbb{R}^4$ equipped with the group law (1.6). We let X_0, \dots, X_4 be the left-invariant vector fields defined by (1.8). In this context H is the vector bundle spanned by X_1, \dots, X_4 , its complex structure \mathcal{J} is such that $\mathcal{J}X_j = X_{2+j}$ and $\mathcal{J}X_{2+j} = -X_j$ for $j = 1, 2$, and L is the vector bundle spanned by X_1 and X_2 . In addition, we equip \mathbb{H}^5 with its Levi metric $g := \theta^2 + d\theta(\cdot, \mathcal{J}\cdot)$, where $\theta = dx^0 + \sum_{j=1}^2 (x^j dx^{2+j} - x^{2+j} dx^j)$ is the standard contact form of \mathbb{H}^5 . With respect to this metric X_0, \dots, X_4 form an orthonormal frame of $T\mathbb{H}^5$.

In the sequel it will be convenient to introduce for $j = 1, 2$ the complex coordinates $z^j = x^j + ix^{2+j}$ and $\bar{z}^j = x^j - ix^{2+j}$, as well as the vector fields,

$$(5.2) \quad Z_j = \frac{1}{\sqrt{2}}(X_j - iX_{2+j}) = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z^j} + iz^{\bar{j}} \frac{\partial}{\partial x^0}\right),$$

$$(5.3) \quad \bar{Z}_j = \overline{Z_j} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \bar{z}^j} - iz^j \frac{\partial}{\partial x^0}\right).$$

Then $\{Z_1, Z_2, \bar{Z}_1, \bar{Z}_2\}$ is a left-invariant orthonormal frame of $H \otimes \mathbb{C}$. Because of the way the CR and real structures are defined in terms of the vector fields X_1, \dots, X_4 , this orthonormal frame is admissible in the sense used in the previous section.

Let $\{\theta^1, \theta^2, \theta^{\bar{1}}, \theta^{\bar{2}}\}$ be the dual coframe of $\{Z_1, Z_2, \bar{Z}_1, \bar{Z}_2\}$. In fact, we can check that $\theta^j = \sqrt{2}dz^j$ and $\theta^{\bar{j}} = \sqrt{2}d\bar{z}^j$ for $j = 1, 2$. Since $\{Z_1, Z_2, \bar{Z}_1, \bar{Z}_2\}$ is an admissible orthonormal frame Eq. (3.19) holds. It actually holds without a $O_H(1)$ remainder term. Indeed, as $\bar{\partial}_H = \sqrt{2} \sum \varepsilon(dz^{\bar{j}}) \bar{Z}_j$ we can check that $\bar{\partial}_H^* = -\frac{1}{\sqrt{2}} \sum \iota(dz^j) Z_j$, where $\iota(dz^j)$ denotes the contraction by $\frac{d}{dz^j}$ (it agrees with that by $\sqrt{2}Z_j$ on $H^* \otimes \mathbb{C}$). Following the lines of the proof of Proposition 3.2 we then see that no remainder terms are involved anymore. Thus,

$$(5.4) \quad Q_L = \sum_{j,k=1,2} \left(\varepsilon(dz^{\bar{j}}) \iota(dz^k) - \iota(dz^j) \varepsilon(dz^{\bar{k}}) \right) (Z_j Z_k + Z_j Z_{\bar{k}}).$$

Next, observe that

$$(5.5) \quad \varepsilon(dz^{\bar{j}}) \iota(dz^k) d\bar{z}^{\bar{l}} = \delta^{k\bar{l}} d\bar{z}^{\bar{j}} \quad \text{and} \quad \iota(dz^j) \varepsilon(dz^{\bar{k}}) = (1 - \delta^{k\bar{l}}) (\delta^{j\bar{k}} d\bar{z}^{\bar{l}} - \delta^{j\bar{l}} d\bar{z}^{\bar{k}}).$$

Using this we can check that, with respect to the frame $\{dz^{\bar{1}}, dz^{\bar{2}}\}$, on $(0, 1)$ -forms Q_L takes the form,

$$(5.6) \quad Q_L = \begin{pmatrix} \Delta_1 - \Delta_2 & T \\ T & \Delta_2 - \Delta_1 \end{pmatrix},$$

where we have set $\Delta_j := Z_j Z_j + Z_j Z_{\bar{j}}$, $j = 1, 2$, and $T := 2(Z_1 Z_2 + Z_1 Z_{\bar{2}})$, and we also have used the fact that $[Z_1, Z_2] = [Z_2, Z_1] = 0$.

Let $\mathcal{F}_0 u := \int_{-\infty}^{\infty} e^{-ix^0 \cdot \xi_0} u dx^0$ denote the Fourier transform with respect to the variable x_0 on \mathcal{S}' . We shall now look at Q_L under \mathcal{F}_0 . Notice this is merely the same as looking at Q_L under the irreducible representations of \mathbb{H}^5 . To this end we shall use the symbol $\hat{\cdot}$ to denote the conjugation by \mathcal{F}_0 . We have

$$(5.7) \quad \hat{Z}_j = \frac{1}{\sqrt{2}} \mathcal{F}_0 \left(\frac{\partial}{\partial z^j} + iz^{\bar{j}} \frac{\partial}{\partial x^0} \right) \mathcal{F}_0^{-1} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial z^j} - z^{\bar{j}} \xi_0 \right),$$

$$(5.8) \quad \hat{\bar{Z}}_j = \frac{1}{\sqrt{2}} \mathcal{F}_0 \left(\frac{\partial}{\partial \bar{z}^j} - iz^j \frac{\partial}{\partial x^0} \right) \mathcal{F}_0^{-1} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial \bar{z}^j} + z^j \xi_0 \right).$$

Using this we can check that $2\hat{Z}_{\bar{j}}\hat{Z}_j$ is equal to

$$(5.9) \quad \left(\frac{\partial}{\partial z^{\bar{j}}} + z^j \xi_0 \right) \left(\frac{\partial}{\partial \bar{z}^j} - z^{\bar{j}} \xi_0 \right) = \frac{\partial^2}{\partial z^{\bar{j}} \partial \bar{z}^j} + \xi_0 \left(-\frac{\partial}{\partial z^{\bar{j}}} z^{\bar{j}} + z^j \frac{\partial}{\partial \bar{z}^j} \right) - z^j z^{\bar{j}} \xi_0^2 \\ = H_j + \xi_0 R_j - \xi_0,$$

where we have set $H_j := \frac{\partial^2}{\partial z^{\bar{j}} \partial \bar{z}^j} - |z^j|^2 \xi_0^2$ and $R_j := z^j \frac{\partial}{\partial \bar{z}^j} - z^{\bar{j}} \frac{\partial}{\partial z^j}$. Similarly,

$$(5.10) \quad 2\hat{Z}_j\hat{\bar{Z}}_j = H_j + \xi_0 R_j + \xi_0.$$

We also have

$$(5.11) \quad \hat{T} = \left(\frac{\partial}{\partial z^1} + z^1 \xi_0 \right) \left(\frac{\partial}{\partial z^2} - z^2 \bar{\xi}_0 \right) + \left(\frac{\partial}{\partial z^1} - z^1 \bar{\xi}_0 \right) \left(\frac{\partial}{\partial z^2} + z^2 \xi_0 \right).$$

Therefore, on $(0, 1)$ -forms we have

$$(5.12) \quad Q_L = \begin{pmatrix} H_1 - H_2 + \xi_0(R_1 - R_2) & \hat{T} \\ \hat{T} & H_2 - H_1 + \xi_0(R_2 - R_1) \end{pmatrix},$$

with \hat{T} given by (5.11).

In the sequel we set $z = (z^1, z^2)$ and we consider the $(0, 1)$ -form,

$$(5.13) \quad \hat{\omega} := \hat{u} dz^{\bar{1}}, \quad \hat{u}(\xi_0, z) := \exp(-|\xi_0||z|^2).$$

Notice that \hat{u} is a ground state for the harmonic oscillators H_j and annihilates the rotation generators R_j , namely,

$$(5.14) \quad H_1 \hat{u} = H_2 \hat{u} = |\xi_0| \hat{u} \quad \text{and} \quad R_1 \hat{u} = R_2 \hat{u} = 0.$$

In addition \hat{u} also annihilates $(\frac{\partial}{\partial z^j} - z^j \bar{\xi}_0) \hat{u}$ for $\xi_0 \geq 0$ and $(\frac{\partial}{\partial z^j} + z^j \xi_0) \hat{u} = 0$ for $\xi_0 \leq 0$, so using 5.11 we see that

$$(5.15) \quad \hat{T} \hat{u} = 0.$$

Combining all this with (5.12) we get

$$(5.16) \quad \hat{Q}_L \hat{\omega} = (H_1 - H_2 + \xi_0(R_1 - R_2)) \hat{u} dz^{\bar{1}} + \hat{T} \hat{u} dz^{\bar{2}} = 0.$$

Next, the inverse transform $u := \mathcal{F}_0^{-1} \hat{u}$ is equal to

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix^0 \cdot \xi_0} e^{-|z|^2 |\xi_0|} d\xi_0 &= \frac{1}{2\pi} \left(\int_0^{\infty} e^{ix^0 \cdot \xi_0} e^{-|z|^2 \xi_0} d\xi_0 + \int_0^{\infty} e^{-ix^0 \cdot \xi_0} e^{-|z|^2 \xi_0} d\xi_0 \right) \\ &= \frac{1}{2\pi} \left(\frac{1}{ix^0 - |z|^2} + \frac{1}{-ix^0 - |z|^2} \right) = \frac{-1}{\pi} \frac{|z|^2}{|x^0|^2 + |z|^4}. \end{aligned}$$

Notice that u is homogeneous of degree -2 with respect to the dilations (1.7). In particular u is singular at the origin.

Set $\omega = \mathcal{F}_0^{-1} \hat{\omega} = u dz^{\bar{1}}$. In view of (5.16) we have

$$(5.17) \quad Q_L \omega = \mathcal{F}_0^{-1} \hat{Q}_L \hat{\omega} = 0.$$

Therefore, we see that, although ω is not smooth at the origin, $Q_L \omega$ is smooth everywhere. This shows that Q_L is not hypoelliptic on $(0, 1)$ -forms.

In fact, the same arguments as above also show that the forms $u dz^1 \wedge dz^{\bar{1}}$ and $u dz^1 \wedge dz^2 \wedge dz^{\bar{1}}$ too annihilate Q_L . Therefore Q_L is not hypoelliptic on $(1, 1)$ -forms or $(2, 1)$ -forms. This shows that this is *only* on $(p, 0)$ -forms and $(p, 2)$ -forms that Q_L is hypoelliptic.

6. THE FREDHOLM INDEX OF Q_L

Throughout this section we assume that M is compact and has finite type.

Set $\Lambda := \Lambda^{*,0} \oplus \Lambda^{*,n}$. A consequence of Proposition 3.2 is that $Q_{L|\Lambda}$ is continuous from $S_2^2(M, \Lambda)$ to $L^2(M, \Lambda)$, where $S_2^2(M, \Lambda)$ is the Folland-Stein weighted Sobolev space defined as in (4.3)–(4.4). By Proposition 4.1 the operator $Q_{L|\Lambda}$ admits a parametrix in the class of singular-integral operators of Rothschild-Stein [RS] with a smoothing remainder terms. In particular, such a parametrix maps continuously $L^2(M, \Lambda)$ to $S_2^2(M, \Lambda)$ and the remainder terms are compact operators on $L^2(M, \Lambda)$

and $S_2^2(M, \Lambda)$. Therefore, seen as a continuous linear mapping from $S_2^2(M, \Lambda)$ to $L^2(M, \Lambda)$, the operator $Q_{L|\Lambda}$ is invertible modulo compact operators, hence is a Fredholm operator.

Next, as Λ is preserved by γ we have the splitting,

$$(6.1) \quad \Lambda = \Lambda^+ \oplus \Lambda^-, \quad \Lambda^\pm := \ker(\gamma|_\Lambda \mp 1).$$

Notice that, as γ is selfadjoint, this splitting is orthogonal. Since the definition of the operator Q_L in (3.2) implies that $\gamma Q_L = -Q_L \gamma$, we see that with respect to the above splitting $Q_{L|\Lambda}$ takes the form,

$$(6.2) \quad Q_{L|\Lambda} = \begin{pmatrix} 0 & Q_L^- \\ Q_L^+ & 0 \end{pmatrix}, \quad Q_L^\pm : C^\infty(M, \Lambda^\pm) \rightarrow C^\infty(M, \Lambda^\mp).$$

Notice that as Q_L is selfadjoint $Q_L^- = (Q_L^+)^*$. We define the index of Q_L to be

$$(6.3) \quad \text{ind } Q_L := \dim \ker Q_L^+ - \dim \ker Q_L^-.$$

Notice that $\text{ind } Q_L$ agrees with the Fredholm index of Q_L^+ in the usual sense, if we regard Q_L as an operator from $S_2^2(M, \Lambda^+)$ to $L^2(M, \Lambda^-)$.

Since Q_L is a geometric operator and, as illustrated by Section 1, there is a handful of geometric examples of CR manifolds carrying real structures, we could have expected the index of Q_L to have some geometric significance. This is not the case, however, because of the following.

Proposition 6.1. *We have $\text{ind } Q_L = 0$.*

Proof. First, let $\omega \in \ker Q_{L|\Lambda^{*,0}}$ and set $\omega^\pm = (1 \pm \gamma)\omega$. Then

$$(6.4) \quad \gamma \omega^\pm = \pm(1 \pm \gamma)\omega = \pm \omega^\pm,$$

so ω^\pm is a section of Λ^\pm . Moreover, as $Q_L \gamma = -\gamma Q_L$ we also have

$$(6.5) \quad Q_L \omega^\pm = Q_L(1 \pm \gamma)\omega = (1 \mp \gamma)Q_L \omega = 0,$$

which shows that ω^\pm belongs to $\ker Q_L^\pm$.

Notice that, as γ maps $\Lambda^{*,0}$ onto $\Lambda^{*,n}$, the fact that $\omega^\pm = \omega \pm \gamma \omega$ implies that ω is the component of ω^\pm in $\Lambda^{*,0}$ and $\gamma \omega$ is its components in $\Lambda^{*,n}$. Anyhow, we see that $\omega^\pm = 0$ if and only if $\omega = 0$. This shows that $1 \pm \gamma$ gives rise to a one-to-one linear map from $\ker Q_{L|\Lambda^{*,0}}$ to $\ker Q_L^\pm$.

Conversely, let $\omega \in \ker Q_L^\pm$. Since $\Lambda^\pm \subset \Lambda^{*,0} \oplus \Lambda^{*,n}$ we can uniquely decompose ω as $\omega = \omega^0 + \omega^n$ where ω^0 is a section of $\Lambda^{*,0}$ and ω^n is a section of $\Lambda^{*,n}$. We have $0 = Q_L \omega = Q_L \omega^0 + Q_L \omega^n$. As Q_L preserves the bidegree, we see that $Q_L \omega^0$ is a section of $\Lambda^{*,0}$, while $Q_L \omega^n$ is a section of $\Lambda^{*,n}$. Thus,

$$(6.6) \quad Q_L \omega^0 = Q_L \omega^n = 0.$$

Furthermore, as ω is a section of Λ^\pm we have

$$(6.7) \quad \omega = \pm \gamma \omega = \pm \gamma \omega^0 + \pm \gamma \omega^n.$$

Since γ maps $(p, 0)$ -forms to $(n-p, 0)$ -forms and (p, n) -forms to $(n-p, 0)$ -forms we see that $\pm \gamma \omega^0$ is the component of ω with values in $\Lambda^{*,n}$, so it must agree with ω^0 . Hence $\omega = (1 \pm \gamma)\omega^0$. Therefore $1 \pm \gamma$ maps $\ker Q_{L|\Lambda^{*,0}}$ onto $\ker Q_L^\pm$.

All this shows that $1 \pm \gamma$ is a linear isomorphism from $\ker Q_{L|\Lambda^{*,0}}$ onto $\ker Q_L^\pm$. This implies that $\ker Q_L^+$ and $\ker Q_L^-$ have same dimension, hence $\text{ind } Q_L = 0$. \square

The above arguments would have broken down if we had considered the restriction of Q_L on $(\frac{n}{2}, \frac{n}{2})$ -forms, instead of that to $(p, 0)$ -forms and (p, n) -forms. However, it is not possible to define a Fredholm index in that case because, as explained in the previous section, the operator Q_L may fail to be hypoelliptic on $(\frac{n}{2}, \frac{n}{2})$ -forms.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, CANADA.
E-mail address: ponge@math.utoronto.ca